

META-STABILITY AND CONDENSED ZERO-RANGE PROCESSES ON FINITE SETS

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ABSTRACT. We propose a definition of meta-stability and obtain sufficient conditions for a sequence of Markov processes on finite state spaces to be meta-stable. In the reversible case, these conditions reduce to estimates of the capacity and the measure of certain meta-stable sets. We prove that a class of condensed zero-range processes with asymptotically decreasing jump rates is meta-stable.

1. INTRODUCTION

The zero-range dynamics is a Markov process which models the evolution of indistinguishable particles on some set S . It can be informally described as follows. Set $S = \Lambda_L$, the one dimensional discrete torus with L points, fix a finite range probability measure p on \mathbb{Z} and a rate function $g : \mathbb{N} \rightarrow \mathbb{R}_+$ such that $g(0) = 0$, $g(k) > 0$ for $k \geq 1$. If there are n particles at a site x of Λ_L , at rate $p(y)g(n)$ one of them jumps from x to $x + y$. The name “zero-range” derives from the observation that, infinitesimally, the interaction is only with respect to those particles at the particular vertex. The case when $g(k)$ is proportional to k describes the situation of completely independent particles.

Denote by η the configurations of the state space \mathbb{N}^{Λ_L} and by $\eta(x)$, $x \in \Lambda_L$, the number of particles at site x for the configuration η . To describe the invariant states of the process, let $Z : [0, \varphi^*) \rightarrow \mathbb{R}_+$ be the partition function given by

$$Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(k)!},$$

where $g(k)! = g(1) \cdots g(k)$, and where φ^* stands for the radius of convergence of Z . An elementary computation shows that the product measures $\{\nu_\varphi : 0 \leq \varphi < \varphi^*\}$ defined by

$$\nu_\varphi(\eta) = \prod_{x \in \Lambda_L} \frac{1}{Z(\varphi)} \frac{\varphi^{\eta(x)}}{g(\eta(x))!}$$

are stationary states for the Markov dynamics described above. The parameter φ is called the fugacity.

The density of particles under the stationary state ν_φ , denoted by $\rho(\varphi)$, is given by

$$\rho(\varphi) = E_{\nu_\varphi}[\eta(x)] = \frac{1}{Z(\varphi)} \sum_{k \geq 0} k \frac{\varphi^k}{g(k)!} = \varphi \partial_\varphi \log Z(\varphi).$$

The density $\rho : [0, \varphi^*) \rightarrow \mathbb{R}_+$ is a strictly increasing function of the fugacity. Denote $\varphi(\rho)$ the inverse function and by $\rho^* = \lim_{\varphi \rightarrow \varphi^*} \rho(\varphi)$ the critical density. By [11,

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Lemma II.3.3], $\rho^* = \infty$ whenever $\varphi^* = \infty$. In contrast, if $\varphi^* < \infty$ both cases can occur.

Assume that $\rho^* < \infty$, i.e., that we have a phase transition. By [11, Lemma II.3.3], $Z(\varphi^*) < \infty$ and the measure ν_{φ^*} is well defined. Therefore, there is a product stationary state if and only if the density is less than or equal to the critical density ρ^* .

To examine the stationary states above the critical density, denote by $\Sigma_{\Lambda_L, N}$, $N \geq 1$, the set of configurations with N particles: $\Sigma_{\Lambda_L, N} = \{\eta \in \mathbb{N}^{\Lambda_L} : \sum_{x \in \Lambda_L} \eta(x) = N\}$. Since the dynamics preserve the total number of particles, if the random walk on Λ_L associated to the transition probability p is irreducible, the sets $\Sigma_{\Lambda_L, N}$ are the ergodic components of the state space. Let $\mu_{L, N}$ denote the unique stationary probability measure on $\Sigma_{\Lambda_L, N}$.

For $\rho \leq \rho^*$, as the total number of site increases to infinity and the density is kept equal to ρ , the equivalence of ensembles [11, Appendix 2] states that the marginals of the measure $\mu_{L, N}$ converges to the measure $\nu_{\varphi(\rho)}$. More precisely, for any $n \geq 1$ and any (k_1, \dots, k_n) in \mathbb{N}^n ,

$$\lim_{L \rightarrow \infty} \mu_{L, [\rho L]} \{\eta : \eta(x_i) = k_i, 1 \leq i \leq n\} = \nu_{\varphi(\rho)} \{\eta : \eta(x_i) = k_i, 1 \leq i \leq n\}.$$

In this formula, $[a]$ stands for the integer part of $a \in \mathbb{R}$.

Above the critical density, in the stationary state, all additional particles concentrate on *one single* site [6, 10, 9, 8]. In the remaining sites, particles are distributed according to the grand canonical measure at the critical density [9, 1]: for $\rho > \rho^*$,

$$\begin{aligned} \lim_{L \rightarrow \infty} \mu_{L, [\rho L]} \{\eta : \eta(x_i) = k_i, 1 \leq i \leq n\} &= \nu_{\varphi(\rho^*)} \{\eta : \eta(x_i) = k_i, 1 \leq i \leq n\}, \\ \lim_{L \rightarrow \infty} \mu_{L, [\rho L]} \left\{ \max_{1 \leq x \leq L} \eta(x) \geq [(\rho - \rho^*)L] \right\} &= 1. \end{aligned}$$

There is a huge physical literature on condensation. In this respect, the zero-range process and similar interacting particle systems have been used to model traffic flow, phase separation dynamics and sandpile models ([7] and references therein).

In this article we examine the time evolution of the site which concentrates all additional particles in the case where the total number of sites is kept fixed, particles are allowed to jump to any site with uniform probability and the total number of particles increases to infinity. We prove that in a proper time scale the position of the site evolves according to a random walk on Λ_L .

To prove this result we first propose a simple analytical definition of metastability to be compared with the pathwise approach introduced in [5] and the spectral definition in [4]. A sequence of Markov processes $\{\eta_t^N : t \geq 0\}$ indexed by a parameter N is said to be meta-stable if the state space E_N can be partitioned in a fixed finite number of sets $\Delta_N, \mathcal{E}_N^1, \dots, \mathcal{E}_N^\kappa$ with the following two properties.

- (a) The time spent in Δ_N in any interval of time $[0, t]$ vanishes as $N \uparrow \infty$.
- (b) Denote by ξ_t^N the trace of the process η_t^N on $E_N \setminus \Delta_N$. For x in $E_N \setminus \Delta_N$, let $\Psi_N(x) = j$ if x belongs to \mathcal{E}_N^j , $1 \leq j \leq \kappa$. The (not necessarily Markovian) process $\Psi_N(\xi_t^N)$ converges to a Markovian process as $N \uparrow \infty$.

If there is a well deeper than all others, as it is the case in the Curie-Weiss model of [5] for some parameters, the asymptotic Markov process has one absorbing state. In the condensed zero-range processes of Subsection 2.4, in contrast, there are

several well at the same maximal depth and no absorbing state appears in the limit.

We obtain sufficient conditions for a sequence of Markov processes on finite state spaces to be meta-stable. In the reversible case, these conditions reduce to estimates on the capacity and on the measure of the meta-stable sets \mathcal{E}_N^j . We then prove that zero range processes associated to a class of decreasing rate functions $g(\cdot)$ are meta-stable.

2. NOTATION AND RESULTS

2.1. Meta-stability. Let $(E_N)_{N \geq 1}$ be a sequence of finite state spaces. For each fixed $N \geq 1$ consider an irreducible E_N -valued Markov process $(\mathbb{P}_\eta^N)_{\eta \in E_N}$ on the path space $D(\mathbb{R}_+, E_N)$ with generator

$$L_N(f)(\eta) = \sum_{\xi \in E_N} \{f(\xi) - f(\eta)\} R_N(\eta, \xi) .$$

Let \mathbb{E}_η^N denote the expectation with respect to \mathbb{P}_η^N . For $t \geq 0$, let η_t^N stand for the projection $\eta_t^N : D(\mathbb{R}_+, E_N) \mapsto E_N$. As usual, we shall also use η_t^N to refer to the Markov Process $(\mathbb{P}_\eta^N)_{\eta \in E_N}$.

Fix an integer $\kappa \geq 1$ such that

$$\kappa < \liminf_{N \rightarrow \infty} |E_N| ,$$

where $|E_N|$ stands for the cardinality of E_N . Let $S = \{1, 2, \dots, \kappa\}$. For each E_N and each $x \in S$ fix a subset $\mathcal{E}_N^x \subseteq E_N$ such that $\mathcal{E}_N^x \cap \mathcal{E}_N^y = \emptyset$ for all $x \neq y$. Let us denote $\mathcal{E}_N = \cup_{x \in S} \mathcal{E}_N^x$ and $\Delta_N = E_N \setminus \mathcal{E}_N$, so that we get the partition

$$E_N = \left(\bigcup_{x \in S} \mathcal{E}_N^x \right) \cup \Delta_N .$$

Let $\Psi_N : \mathcal{E}_N \mapsto S$ be given by

$$\Psi_N(\eta) = \sum_{x \in S} x \mathbf{1}\{\eta \in \mathcal{E}_N^x\} .$$

In order to introduce the notion of meta-stability we need to recall the definition of the trace of a Markov process on a subset of the state space. Let F be a subset of E_N . For each $t \geq 0$, let $\mathcal{T}_t^F : D(\mathbb{R}_+, E_N) \mapsto \mathbb{R}_+$ be the time the process stayed in the set F in the interval $[0, t]$:

$$\mathcal{T}_t^F(\omega) := \int_0^t \mathbf{1}\{\omega(s) \in F\} ds$$

and let \mathcal{S}_t^F be the generalized inverse of \mathcal{T}_t^F :

$$\mathcal{S}_t^F(\omega) := \sup\{s \geq 0 : \mathcal{T}_s^F(\omega) \leq t\} .$$

For any path $\omega \in D(\mathbb{R}_+, E_N)$ such that $\omega(0) \in F$, we define a new path $\mathcal{R}^F(\omega) \in D(\mathbb{R}_+, F)$ by $\mathcal{R}^F(\omega)(t) = \omega(\mathcal{S}_t^F(\omega))$. It is well known that, if for each $\eta \in F$ we denote

$$\mathbb{P}_\eta^F := \mathbb{P}_\eta^N \circ (\mathcal{R}^F)^{-1} ,$$

then $(\mathbb{P}_\eta^F)_{\eta \in F}$ is a strong Markov process with path space $D(\mathbb{R}_+, F)$ (see [3, V.2.11]). This Markov process is called the trace of the Markov process η_t^N on F . Let η_t^F denote the projection $\eta_t^F : D(\mathbb{R}_+, F) \mapsto F$, for $t \geq 0$. η_t^F will also be used to refer to the trace of η_t^N on F .

Consider the trace of η_t^N on \mathcal{E}_N , referred to as $\eta_t^{\mathcal{E}_N}$. For each $\eta \in \mathcal{E}_N$ denote by \mathbb{Q}_η^N the measure on $D(\mathbb{R}_+, S)$ induced by the process $X_t^N := \Psi_N(\eta_t^{\mathcal{E}_N})$ when $\eta_t^{\mathcal{E}_N}$ starts at η .

Definition 2.1 (Meta-stability). *The sequence of Markov processes $(\eta_t^N)_{N \geq 1}$ is said to be meta-stable with meta-states given by $(\mathcal{E}_N^x)_{x \in S}$ and limit given by a Markov process $(\mathbb{Q}_x)_{x \in S}$ on $D(\mathbb{R}_+, S)$ if*

- (A) *For any $x \in S$ and any sequence $\eta_N \in \mathcal{E}_N^x$, $N \geq 1$, the sequence $\mathbb{Q}_{\eta_N}^N$ converges to \mathbb{Q}_x .*
- (B) *For any $t > 0$,*

$$\lim_{N \rightarrow \infty} \max_{\eta \in \mathcal{E}_N} \mathbb{E}_\eta^N [\mathcal{T}_t^{\Delta_N}] = 0.$$

Condition (B) states that η_t^N spends a vanishing period of time on the subset Δ_N . Condition (A) states that, after cutting off the negligible time η_t^N spends on Δ_N , the asymptotic evolution of the trace process is described by a Markov process on S .

In fact, if the meta-stability behaviour of η_t^N is assumed, such cut off is no more necessary to observe the asymptotic evolution. More precisely, consider the stochastic process \hat{X}_t^N with state space S defined as

$$\hat{X}_t^N = \Psi_N(\eta_{\sigma(t)}^N),$$

where $\sigma(t) := \sup\{s \leq t : \eta_s^N \in \mathcal{E}_N\}$. Note that \hat{X}_t^N is well defined whenever η_t^N starts at a point in \mathcal{E}_N . Denote by $\hat{\mathbb{Q}}_\eta^N$ the measure on $D(\mathbb{R}_+, S)$ induced by \hat{X}_t^N when η_t^N starts at $\eta \in \mathcal{E}_N$.

Proposition 2.2. *If the sequence of Markov processes $(\eta_t^N)_{N \geq 1}$ is meta-stable with meta-states given by $(\mathcal{E}_N^x)_{x \in S}$ and limit given by the Markov process $(\mathbb{Q}_x)_{x \in S}$, then for any $x \in S$ and any sequence $\eta_N \in \mathcal{E}_N^x$, $N \geq 1$, the sequence $\hat{\mathbb{Q}}_{\eta_N}^N$ converges to \mathbb{Q}_x .*

We postpone the proof of this proposition to Section 4.

2.2. Entry times and meta-stability. The purpose of this subsection is to provide sufficient conditions on the sequence of Markov processes η_t^N to ensure a non-trivial meta-stability behavior of the dynamics.

To state such conditions we shall use entry times. For a subset G of E_N we denote by $H_G : D(\mathbb{R}_+, E_N) \mapsto \mathbb{R}_+$ the entry time in G defined as

$$H_G(\omega) := \inf\{t \geq 0 : \omega(t) \in G\}, \quad (2.1)$$

with the convention that $H_\eta = H_{\{\eta\}}$ when G is a singleton $\{\eta\}$. By abuse of notation we also denote by $H_G : D(\mathbb{R}_+, F_N) \mapsto [0, +\infty]$ the entry time in $G \subseteq F_N$ for any F_N subset of E_N .

Let $(R_N^\mathcal{E}(\eta, \xi) : \eta, \xi \in \mathcal{E}_N)$ be the transition rates of $\eta_t^{\mathcal{E}_N}$ and let $L_N^\mathcal{E}$ be the corresponding generator:

$$L_N^\mathcal{E} f(\eta) = \sum_{\xi \in \mathcal{E}_N} \{f(\xi) - f(\eta)\} R_N^\mathcal{E}(\eta, \xi).$$

For each pair $x, y \in S$, $x \neq y$ and $\eta \in \mathcal{E}_N^x$ we define

$$R_N^\mathcal{E}(\eta, y) := \sum_{\xi \in \mathcal{E}_N^y} R_N^\mathcal{E}(\eta, \xi).$$

Recall that η_t^N is supposed to be irreducible. Denote by ν_N its unique invariant probability measure. Let $r_N(x, y)$ be the expectation of $R_N^\mathcal{E}(\cdot, y)$ with respect to ν_N conditioned on \mathcal{E}_N^x :

$$r_N(x, y) := \frac{1}{\nu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} R_N^\mathcal{E}(\eta, y) \nu_N(\eta) .$$

We shall require that the following limit exists

$$r(x, y) := \lim_{N \rightarrow \infty} r_N(x, y) . \quad (\text{C1})$$

for all $x, y \in S$. The limits $(r(x, y) : x, y \in S)$ become the transition rates of the Markov process limit in condition (A).

The Markov property for the limit of X_t^N follows from a suitable ergodic assumption for the dynamics on each \mathcal{E}_N^x . For each set \mathcal{E}_N^x , choose a site ξ_N^x and let $M_N = \{\xi_N^x : x \in S\}$ be the meta-stable sites. We shall assume that for each $x \in S$, the time it takes for η_t^N to visit the meta-stable site ξ_N^x , once it enters \mathcal{E}_N^x is of sufficiently small order. More precisely, we shall assume that

$$\lim_{N \rightarrow \infty} \left(\max_{\eta \in \mathcal{E}_N^x} R_N^\mathcal{E}(\eta, y) \right) \left(\max_{\eta \in \mathcal{E}_N^x} \mathbb{E}_\eta^N[H_{\xi_N^x}] \right) = 0 , \quad (\text{C2})$$

for all $x, y \in S$.

In this assumption we could replace $\mathbb{E}_\eta^N[H_{\xi_N^x}]$ by the smaller quantity $\mathbb{E}_\eta^\mathcal{E}[H_{\xi_N^x}]$. The difference, however, is the time spent in Δ_N which we require to vanish in condition (B).

For condition (B), we need to regard, for each $x \in S$, the trace of η_t^N on $E_N \setminus \mathcal{E}_N^x$, which will be denoted by $(\mathbb{P}_\eta^{N,x})_{\eta \in E_N \setminus \mathcal{E}_N^x}$. Denote by $\mathbb{E}_\eta^{N,x}$ the expectation with respect to $\mathbb{P}_\eta^{N,x}$. For each $x \in S$, let us denote by $\partial_x \Delta_N$ the set of sites in Δ_N which may be attained from a site in \mathcal{E}_N^x after a jump.

$$\partial_x \Delta_N := \{ \xi \in \Delta_N : R_N(\eta, \xi) > 0 \text{ for some } \eta \in \mathcal{E}_N^x \} .$$

Let also $\tilde{\mathcal{E}}_N^x$ stands for $\mathcal{E}_N \setminus \mathcal{E}_N^x$. We shall require that

$$\lim_{N \rightarrow \infty} \max_{\eta \in \partial_x \Delta_N} \mathbb{E}_\eta^{N,x}[H_{\tilde{\mathcal{E}}_N^x}] = 0 \quad (\text{C3})$$

for all $x \in S$, where the maximum is equal to zero, if $\partial_x \Delta_N$ is empty. In particular, (C3) is automatically satisfied if $\partial_x \Delta_N$ is empty for all $N \geq 0$.

We now state our first main result in this article.

Theorem 2.3. *A sequence of Markov processes satisfying conditions (C1) – (C3) is meta-stable in the sense of Definition 2.1, with meta-states given by $(\mathcal{E}_N^x)_{x \in S}$ and limit given by the Markov process $\{\mathbb{Q}_x\}_{x \in S}$ with transition rates $(r(x, y) : x, y \in S)$.*

It is important to notice that in this theorem we do not exclude the possibility that $\nu_N(\mathcal{E}_N^x)$ vanishes as $N \uparrow \infty$ for some $x \in S$. The proof of this theorem is given in Section 2.3.

2.3. The reversible case, potential theory. In this subsection, we restrict ourselves to the case where ν_N is a reversible probability measure for all $N \geq 1$.

Our aim is to state conditions concerning the generator L_N and the invariant measure ν_N which ensure that Theorem 2.3 is in force. We then need good estimates for the mean of entry times. As we shall see in Section 4, in the reversible case,

the mean of an entry time has a simple expression involving capacities, which are defined as follows. For two disjoint subsets F, G of E_N define

$$\mathcal{B}_N(F, G) := \{f : E_N \mapsto \mathbb{R} : f(\eta) = 1 \ \forall \eta \in F \text{ and } f(\xi) = 0 \ \forall \xi \in G\}.$$

We denote by $f_{F,G} : E_N \mapsto \mathbb{R}$ the function in $\mathcal{B}_N(F, G)$ defined as

$$f_{F,G}(\eta) := \mathbb{P}_\eta^N[H_F < H_G].$$

An elementary computation shows that $f_{F,G}$ solves

$$\begin{cases} (L_N f_{F,G})(\eta) = 0 & \eta \in E \setminus (F \cup G), \\ f_{F,G}(\eta) = 1 & \eta \in F, \\ f_{F,G}(\eta) = 0 & \eta \in G. \end{cases}$$

The maximum principle guarantees that there is a unique solution of this equation given by $f_{F,G}$. Let $\nu_N(\cdot)$ stand for the expectation with respect to ν_N and let $\langle \cdot, \cdot \rangle_{\nu_N}$ stand for the scalar product in $L^2(\nu_N)$. Denote by D_N the Dirichlet form associated to the generator L_N :

$$D_N(f) := \langle -L_N f, f \rangle_{\nu_N}$$

for every $f : E_N \mapsto \mathbb{R}$. The capacity of two disjoint subsets F, G of E_N is defined as

$$\text{cap}_N(F, G) := \inf \{ D_N(f) : f \in \mathcal{B}_N(F, G) \}.$$

Observe that $\text{cap}_N(F, G) = \text{cap}_N(G, F)$. If F or G are equal to an unitary set $\{\eta\}$ in any notation just introduced, then we shall write η instead of $\{\eta\}$. It is well known that, since ν_N is supposed to be reversible,

$$\text{cap}_N(F, G) = D_N(f_{F,G})$$

(see Theorem II.6.1 in [12]).

We now state the hypotheses we shall use to prove that condition **(C3)** is in force. Choose for each $x \in S$ and $N \geq 1$, a suitable site $\zeta_N^x \in \partial_x \Delta_N$, whenever $\partial_x \Delta_N$ is non-empty. We require that

$$\lim_{N \rightarrow \infty} \frac{\nu_N(\Delta_N)}{\text{cap}_N(\zeta_N^x, \tilde{\mathcal{E}}_N^x)} = 0. \quad (\text{H2})$$

Of course, **(H2)** follows from the stronger condition

$$\lim_{N \rightarrow \infty} \nu_N(\Delta_N) = 0 \quad \text{and} \quad \inf_{N \geq 1} \text{cap}_N(\zeta_N^x, \tilde{\mathcal{E}}_N^x) > 0, \quad (\text{H2}')$$

where the infimum in the right hand side is carried over all $N \geq 1$ such that $\partial_x \Delta_N$ is non-empty. In case that $\partial_x \Delta_N$ is empty for all $N \geq 1$ then the right hand side is supposed to be automatically satisfied.

Let $\overline{\mathcal{E}}_N^x$ stand for the set $\mathcal{E}_N^x \cup \partial_x \Delta_N$. In addition, for each pair x, y of different elements of S we denote by $\partial \mathcal{E}_N^{x,y}$ the support of $R_N^\mathcal{E}(\cdot, y)$ on \mathcal{E}_N^x :

$$\partial \mathcal{E}_N^{x,y} := \{\eta \in \mathcal{E}_N^x : R_N^\mathcal{E}(\eta, y) > 0\}.$$

As our ergodic hypotheses, we shall assume that for any pair $x, y \in S$ such that $x \neq y$ there holds

$$\lim_{N \rightarrow \infty} \min \{ \text{cap}_N(\eta, \xi_N^x) : \eta \in \overline{\mathcal{E}}_N^x \} \min \{ \nu_N(\eta) : \eta \in \partial \mathcal{E}_N^{x,y} \} = \infty, \quad (\text{H3})$$

where the second minimum is equal to 1 if $\partial \mathcal{E}_N^{x,y}$ is empty.

We now state our second main theorem that assures a meta-stable behaviour in the reversible setting. Its proof is postponed to Section 4.

Theorem 2.4. *A sequence of reversible Markov processes satisfying conditions (C1), (H2), (H3) is meta-stable in the sense of Definition 2.1, with meta-states given by $(\mathcal{E}_N^x)_{x \in S}$ and limit given by the Markov process $(\mathbb{Q}_x)_{x \in S}$ with transition rates $(r(x, y) : x, y \in S)$.*

2.4. Meta-stable behavior of condensed zero-range processes. Fix an integer $\kappa \geq 2$ and the set of sites $S = \{1, 2, \dots, \kappa\}$. Denote by \mathbb{N} the set of nonnegative integer numbers. For each $N \geq 1$, consider the set of configurations

$$E_N := \{\eta \in \mathbb{N}^S : \sum_{x \in S} \eta(x) = N\}$$

where $\eta(x)$ stands for the number of particles at the site $x \in S$. Fix $\alpha > 1$ and let $g : \mathbb{N} \setminus \{0\} \mapsto \mathbb{R}$ be given by $g(1) = 1$ and

$$g(n) = \left(\frac{n}{n-1}\right)^\alpha, \quad \forall n \geq 2$$

in such a way that $g(1)g(2) \dots g(n) = n^\alpha$ for all $n \geq 1$. Notice that $(g(n) : n \geq 2)$ is a strictly decreasing sequence and that $g(n) \rightarrow 1$ as $n \rightarrow \infty$. On each E_N consider the zero range process whose generator L_N is given by

$$L_N f(\eta) = \sum_{x, y \in S} g(\eta(x)) \mathbf{1}\{\eta(x) > 0\} \{f(\sigma^{x,y}\eta) - f(\eta)\}$$

for $f : E_N \mapsto \mathbb{R}$. Here $\sigma^{x,y}\eta$ is the configuration obtained from η letting a particle jump from x to y :

$$\sigma^{x,y}\eta(z) = \begin{cases} \eta(x) - 1 & \text{if } z = x, \\ \eta(y) + 1 & \text{if } z = y, \\ \eta(z) & \text{otherwise,} \end{cases}$$

so $\sigma^{x,y}\eta$ is well defined whenever $\eta(x) > 0$.

For each $N \geq 1$, we denote by ν_N the unique invariant probability measure corresponding to L_N . It is easy to see that each ν_N is a reversible measure and can be written as

$$\nu_N(\eta) = \frac{1}{Z_{N,\kappa}} \frac{N^\alpha}{\prod_{x \in S} p(\eta(x))}, \quad \forall \eta \in E_N$$

where $p(0) = 1$, $p(n) = n^\alpha$ if $n \geq 1$ and $Z_{N,\kappa} > 0$ is a normalizing constant. It is not difficult to prove that for each $\kappa \geq 2$, there exists a constant $\delta_\kappa > 0$ such that

$$\delta_\kappa < Z_{N,\kappa} < \delta_\kappa^{-1} \quad (2.2)$$

for every $N \geq 1$.

Let $(\ell_N)_{N \geq 1}$ be a sequence of positive integers such that

$$\lim_{N \rightarrow \infty} N^{-1} \ell_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \ell_N = \infty.$$

This sequence determines the subsets of E_N :

$$\mathcal{E}_N^x := \{\eta \in \mathbb{N}^S; \eta(x) \geq N - \ell_N\}, \quad x \in S.$$

Since $N^{-1} \ell_N \rightarrow 0$, for N large enough we have $\mathcal{E}_N^x \cap \mathcal{E}_N^y = \emptyset$ for all $x \neq y$ and $E_N \setminus (\cup_{x \in S} \mathcal{E}_N^x) \neq \emptyset$. Let \mathcal{E}_N stand for $\cup_{x \in S} \mathcal{E}_N^x$ and Δ_N stand for $E_N \setminus \mathcal{E}_N$.

To establish the time-scale at which the zero-range process exhibit a non-trivial meta-stable behavior, let $(\theta_N)_{N \geq 1}$ be a sequence of positive numbers. For each $N \geq$

1, let η_t^N stand for the zero range process L_N speeded-up by θ_N . Let $(R_N(\eta, \xi) : \eta, \xi \in E_N)$ denote the transition rates of η_t^N :

$$\theta_N L_N f(\eta) = \sum_{\xi \in E_N} \{f(\xi) - f(\eta)\} R_N(\eta, \xi) .$$

We shall choose the time scale by examining condition **(C1)**.

Let $\text{cap}(\cdot, \cdot)$ be the capacity with respect to the generator L_N . We choose the sequence $(\theta_N)_{N \geq 1}$ as

$$\theta_N = \frac{1}{\text{cap}(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x)} , \quad \text{for } N \geq 1 . \quad (2.3)$$

Since particles jump to any site with equal probability, the right hand side does not depend on $x \in S$. We prove in Lemma 3.2 below that there exists a constant $C_0 > 0$ such that

$$\frac{1}{C_0} < \frac{\theta_N}{N^{1+\alpha}} < C_0$$

for all $N \geq 1$.

Recall that for $x, y \in S$, $x \neq y$ we denote

$$r_N(x, y) = \frac{1}{\nu_N(\mathcal{E}_N^x)} \sum_{\substack{\eta \in \mathcal{E}_N^x \\ \xi \in \mathcal{E}_N^y}} R_N(\eta, \xi) \nu_N(\eta) .$$

Notice that $r_N(x, y)$ does not depend on $x, y \in S$. By formula (c) in Lemma 5.4,

$$\sum_{y \in S \setminus \{x\}} r_N(x, y) = \kappa , \quad \forall x \in S$$

and so, $r_N(x, y) = \kappa(\kappa - 1)^{-1}$ for all $N \geq 1$ and all $x, y \in S$, $x \neq y$. In particular, condition **(C1)** is trivially satisfied.

Theorem 2.5. *Let $(\theta_N)_{N \geq 1}$ be given by (2.3). If $\ell_N \rightarrow \infty$ and*

$$\lim_{N \rightarrow \infty} \frac{(\ell_N)^{2\alpha(\kappa-1)+1}}{N^{1+\alpha}} = 0 , \quad (2.4)$$

then the sequence of zero-range processes η_t^N is meta-stable in the sense of Definition 2.1, with meta-states given by $(\mathcal{E}_N^x)_{x \in S}$. The meta-stable behaviour is described by a Markov process on S with transition rates $r(x, y) = \kappa(\kappa - 1)^{-1}$ for any $x, y \in S$, $x \neq y$.

3. CONDENSED ZERO-RANGE PROCESSES

We show in this section that the condensed zero-range processes introduced in Section 1 present a meta-stable behavior in the sense of Definition 2.1. We start with a generalization of zero-range processes on two sites. Throughout this section we use systematically the notation introduced in the previous section.

3.1. Birth and death process. Fix $a < b$ in \mathbb{R} and consider a nonnegative smooth function $H : [a, b] \rightarrow \mathbb{R}_+$. Assume that H vanishes only at a finite number of points denoted by $a_1 < a_2 < \dots < a_m$:

$$H(x) = 0 \quad \text{if and only if} \quad x \in \{a_1, \dots, a_m\}.$$

We do not exclude the possibility that H vanishes at the boundary points a, b .

For each $i = 1, \dots, m$, assume that there exist a neighborhood V_{a_i} of a_i and $\alpha_i > 0$ such that

$$H(x) = |x - a_i|^{\alpha_i} \quad \text{for all } x \in V_{a_i},$$

and that $V_{a_i} \cap V_{a_j} = \emptyset$ for $i \neq j$. Let $\alpha = \max\{\alpha_i : 1 \leq i \leq m\}$. Assume that $\alpha > 1$ and that there are at least two exponents α_i equal to α :

$$\kappa := |\{i : \alpha_i = \alpha\}| \geq 2,$$

where $|A|$ indicates the cardinality of a finite set A . Denote by $b_1 < b_2 < \dots < b_\kappa$ the elements of $\{a_1, \dots, a_m\}$ whose associated exponents are α .

The definition of the state space of the birth and death process requires some notation. Fix $N \geq 1$, $1 \leq i \leq m-1$, and let

$$\begin{aligned} k_0^N &= \min \left\{ k \geq 0 : a > a_1 - (k+1)/N \right\}, \\ k_i^N &= \min \left\{ k \geq 0 : a_i + (k+1)/N > a_{i+1} - (k+1)/N \right\}, \\ k_m^N &= \min \left\{ k \geq 0 : a_m + (k+1)/N > b \right\}. \end{aligned}$$

For $1 \leq i \leq m-1$, set

$$\begin{aligned} G_{N,0} &= \{a_1 - k_0^N/N, \dots, a_1 - 1/N, a_1\}, \\ G_{N,i} &= \{a_i + 1/N, a_i + 2/N, \dots, a_i + k_i^N/N, a_{i+1} - k_i^N/N, \dots, a_{i+1}\}, \\ G_{N,m} &= \{a_m + 1/N, a_m + 2/N, \dots, a_m + k_m^N/N\} \end{aligned}$$

and let the state space E_N be $\bigcup_{i=0}^m G_{N,i}$. Note that the exact definition of E_N is not important for the meta-stability behavior discussed in this section. The elements of E_N are denoted by the letters x, y, z . Two points $x < y$ are said to be neighbors in E_N if there is no z in E_N such that $x < z < y$.

Let ν_N be the probability measure on E_N defined by

$$\nu_N(x) = \begin{cases} \frac{1}{Z_N} \frac{1}{H(x)} & \text{if } x \notin \{a_1, \dots, a_m\}, \\ \frac{1}{Z_N} N^{\alpha_i} & \text{if } x = a_i \text{ for some } 1 \leq i \leq m. \end{cases}$$

In this formula Z_N is a normalizing constant. An elementary computation shows that

$$\lim_{N \rightarrow \infty} \frac{Z_N}{N^\alpha} = \sum_{i=1}^{\kappa} \left\{ 1 + \sigma_i \sum_{k \geq 1} \frac{1}{k^\alpha} \right\}. \quad (3.1)$$

where $\sigma_i = 1$ if $b_i \in \{a, b\}$ and $\sigma_i = 2$ otherwise. In particular, if we denote

$$m(b_i) = 1 + \sigma_i \sum_{k \geq 1} \frac{1}{k^\alpha}, \quad \text{for } i = 1, \dots, \kappa$$

then

$$\lim_{N \rightarrow \infty} \nu_N(b_i) = \left\{ \sum_{i=1}^{\kappa} m(b_i) \right\}^{-1} > 0 \quad (3.2)$$

for every $1 \leq i \leq \kappa$.

We now fix a convenient sequence of neighborhoods around each point b_i , $i = 1, \dots, \kappa$. Let $(\ell_N)_{N \geq 1}$ be a sequence of positive integers, both small for the macroscopic scale and large for the microscopic scale:

$$\lim_{N \rightarrow \infty} N^{-1} \ell_N = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \ell_N = \infty.$$

For each b_i , $1 \leq i \leq \kappa$, we define

$$\mathcal{E}_N^i := E_N \cap \left[b_i - \frac{\ell_N}{N}, b_i + \frac{\ell_N}{N} \right].$$

Notice that, because $N^{-1} \ell_N \rightarrow 0$, for N large enough we have $\mathcal{E}_N^i \subseteq V_{b_i}$ for every $1 \leq i \leq \kappa$. In particular, for N large enough, $\mathcal{E}_N^i \cap \mathcal{E}_N^j = \emptyset$ for all $i \neq j$. Moreover, since $\ell_N \rightarrow \infty$,

$$\lim_{N \rightarrow \infty} \nu_N(\Delta_N) = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \nu_N(\mathcal{E}_N^i) = \frac{m(b_i)}{\sum_{j=1}^{\kappa} m(b_j)} \quad (3.3)$$

for all $1 \leq i \leq \kappa$, where $\Delta_N := E_N \setminus (\cup_{i=1}^{\kappa} \mathcal{E}_N^i)$.

In view of the theory presented in Section 2, this concentration of the measure on small macroscopic subsets suggests that a Markov process on E_N , reversible with respect to ν_N , is a good candidate to exhibit a meta-stable behavior with $\{b_1, \dots, b_{\kappa}\}$ as meta-stable points.

Fix a positive function $\lambda : [a, b] \rightarrow \mathbb{R}_+$ bounded above and below by a strictly positive constant:

$$0 < \delta \leq \lambda(x) \leq \delta^{-1}.$$

This assumption is not really needed but we do not seek optimal assumptions in this subsection. Consider a birth and death process $\{Z_t^N : t \geq 0\}$ on E_N with rates given by

$$R_N(x, y) = \begin{cases} N^{1+\alpha} \lambda(x) & \text{if } x > y, \\ N^{1+\alpha} \lambda(y) \frac{\nu_N(y)}{\nu_N(x)} & \text{if } x < y, \end{cases}$$

provided x and y are neighbors in E_N . The process is of course reversible with respect to the measure ν_N . As we shall see in the proof of the next result $N^{1+\alpha}$ is the correct scale to observe a non-trivial meta-stable behaviour.

Theorem 3.1. *Assume that $(\ell_N)_{N \geq 1}$ is chosen such that $\ell_N \rightarrow \infty$ and*

$$\lim_{N \rightarrow \infty} \frac{(\ell_N)^{2\alpha+1}}{N^{\alpha+1}} = 0. \quad (3.4)$$

Then, $(Z_t^N)_{N \geq 1}$ is meta-stable in the sense of Definition 2.1, with meta-states given by $(\mathcal{E}_N^i)_{1 \leq i \leq \kappa}$. The meta-stable behavior is described by a Markov process on $\{1, \dots, \kappa\}$ with rates given by

$$r(i, i+1) = \frac{1}{m(b_i)} \frac{1}{\int_{b_i}^{b_{i+1}} \{H(u)/\lambda(u)\} du},$$

$$r(i+1, i) = \frac{1}{m(b_{i+1})} \frac{1}{\int_{b_i}^{b_{i+1}} \{H(u)/\lambda(u)\} du},$$

for $1 \leq i < \kappa$.

Proof. We shall verify each hypothesis of Theorem 2.4. Let $\xi_N^i = b_i$ for $1 \leq i \leq \kappa$. The first condition of **(H2')** has been derived in (3.3).

To check the second condition of **(H2')** and **(H3)**, we take advantage from the one-dimensional setting to get explicit expressions for capacities. Let $x^N < y^N$ be points in E_N . Recall that $\text{cap}(x^N, y^N) = D_N(f_{x^N, y^N})$ where $f_{x^N, y^N} : E_N \mapsto \mathbb{R}$ solves the equation $L_N f_{x^N, y^N}(z) = 0$ for $z \notin \{x^N, y^N\}$ with boundary conditions $f_{x^N, y^N}(x^N) = 1$ and $f_{x^N, y^N}(y^N) = 0$. An elementary computation gives that $f(z) = 1$ for $z \leq x^N$, $f(z) = 0$ for $z \geq y^N$ and

$$f(z + 1/N) - f(z) = \frac{\{\nu_N(z)R_N(z, z + 1/N)\}^{-1}}{\sum_{z=x^N}^{y^N-1/N} \{\nu_N(z)R_N(z, z + 1/N)\}^{-1}}$$

for $z \in E_N \cap [x^N, y^N[$. Hence,

$$\text{cap}(x^N, y^N) = \frac{1}{\sum_{z=x^N}^{y^N-1/N} \{\nu_N(z)R_N(z, z + 1/N)\}^{-1}}. \quad (3.5)$$

In last two formulae, there is a slight abuse of notation since E_N is not the set $\{z/N : z \in \mathbb{Z} \cap [aN, bN]\}$, but the meaning is clear. In particular, it follows from (3.5) that, if $x^N \rightarrow a'$ and $y^N \rightarrow b'$ for some $0 \leq a' < b' \leq 1$ then

$$\lim_{N \rightarrow \infty} \text{cap}(x^N, y^N) = \frac{1}{\sum_{i=1}^{\kappa} m(b_i)} \left\{ \int_{a'}^{b'} \{H(u)/\lambda(u)\} du \right\}^{-1} > 0. \quad (3.6)$$

We are now in a position to check the second condition of hypothesis **(H2')**. For each $1 \leq i \leq \kappa$ and $N \geq 1$, we choose the point $c_i^N \in \partial_i \Delta_N$ as $c_i^N = b_i - (\ell_N + 1)/N$ if $b_i = b$ and $c_i^N = b_i + (\ell_N + 1)/N$ otherwise. Taking advantage of the one-dimensional setting, it is not difficult to show that $\text{cap}(c_i^N, \bar{\mathcal{E}}_N^i)$ is equal to

$$\mathbf{1}\{i < \kappa\} \text{cap}(c_i^N, b_{i+1} - \ell_N/N) + \mathbf{1}\{1 < i\} \text{cap}(c_i^N, b_{i-1} + \ell_N/N),$$

for $1 \leq i \leq \kappa$. Therefore, hypotheses **(H2')** follows from (3.6).

Recall definition of subsets $\partial \mathcal{E}_N^{i,j}$, $\bar{\mathcal{E}}_N^i$ and $\partial_i \Delta_N$ of E_N introduced in the previous section. To check hypotheses **(H3)** notice that in our model,

$$\bar{\mathcal{E}}_N^i = E_N \cap [b_i - (\ell_N + 1)/N, b_i + (\ell_N + 1)/N] \quad \text{for } 1 \leq i \leq \kappa.$$

Observe that for any $1 \leq i \leq \kappa$ and N large enough, $H(x) = |x - b_i|^\alpha$ for all $x \in \bar{\mathcal{E}}_N^i$. In consequence, by using (3.5), it is easy to see that there exists a positive constant C_0 depending only on λ such that

$$\text{cap}(x, b_i) \geq \frac{C_0 N^{2\alpha+1}}{Z_N(\ell_N + 1)^{\alpha+1}}, \quad (3.7)$$

for any $1 \leq i \leq \kappa$ and $x \in \bar{\mathcal{E}}_N^i$. In particular, by (3.1),

$$\lim_{N \rightarrow \infty} \min \{ \text{cap}(x, b_i) : x \in \bar{\mathcal{E}}_N^i \} = \infty$$

since $N^{-1}\ell_N \rightarrow 0$. Moreover, for any two points $i \neq j$ in $\{1, \dots, \kappa\}$, $\partial \mathcal{E}_N^{i,j} = \{b_i + (j - i)\ell_N/N\}$ if $|i - j| = 1$ and $\partial \mathcal{E}_N^{i,j} = \emptyset$ otherwise. Therefore, in view of assumption (3.4), an elementary calculation shows that hypotheses **(H3)** is in force.

It remains to check hypotheses **(C1)**. Notice that, in our model, $r_N(i, j) = 0$ for $|i - j| > 1$. For G_1, G_2 disjoint subsets of \mathcal{E}_N , let $r_N(G_1, G_2)$ be given by

$$r_N(G_1, G_2) := \frac{1}{\nu_N(G_1)} \sum_{\substack{x \in G_1 \\ y \in G_2}} R_N^\mathcal{E}(x, y) \nu_N(x)$$

in such a way that, for $i \neq j$, $r_N(\mathcal{E}_N^i, \mathcal{E}_N^j)$ coincides with $r_N(i, j)$. Fix an arbitrary $1 \leq i < \kappa$. Observe that letting $G_1 = \cup_{j \leq i} \mathcal{E}_N^j$, and $G_2 = \cup_{j > i} \mathcal{E}_N^j$ we have

$$\begin{aligned} \nu_N(\mathcal{E}_N^i) r_N(i, i+1) &= \nu_N(b_i + \frac{\ell_N}{N}) R_N^\mathcal{E}(b_i + \frac{\ell_N}{N}, b_{i+1} - \frac{\ell_N}{N}) \\ &= \nu_N(G_1) r_N(G_1, G_2). \end{aligned}$$

Therefore, by using formula (c) of Lemma 5.4 for $r_N(G_1, G_2)$,

$$\begin{aligned} r_N(i, i+1) &= \frac{\nu_N(G_1) r_N(G_1, G_2)}{\nu_N(\mathcal{E}_N^i)} = \frac{\text{cap}(G_1, G_2)}{\nu_N(\mathcal{E}_N^i)} \\ &= \frac{\text{cap}(b_i + \ell_N/N, b_{i+1} - \ell_N/N)}{\nu_N(\mathcal{E}_N^i)}. \end{aligned}$$

Analogously, we obtain

$$r_N(i, i-1) = \frac{\text{cap}(b_i - \ell_N/N, b_{i-1} + \ell_N/N)}{\nu_N(\mathcal{E}_N^i)}$$

for any $1 < i \leq \kappa$. Therefore, by (3.6) and (3.3),

$$\lim_{N \rightarrow \infty} r_N(i, i+1) = \frac{1}{m(b_i)} \left\{ \int_{b_i}^{b_{i+1}} \{H(u)/\lambda(u)\} du \right\}^{-1}$$

and

$$\lim_{N \rightarrow \infty} r_N(i+1, i) = \frac{1}{m(b_{i+1})} \left\{ \int_{b_i}^{b_{i+1}} \{H(u)/\lambda(u)\} du \right\}^{-1}$$

for any $1 \leq i < \kappa$. We are done. \square

3.2. Condensed zero-range processes. We prove in this subsection Theorem 2.5. The case $\kappa = 2$ is a particular case of the birth and death processes considered in the previous section. Indeed, since the total number of particles is settled to be equal to N , the number of particles on the first site, $\eta(1)$, is a Markov process on $\{0, 1, \dots, N\}$. The density of particles speeded up by $N^{1+\alpha}$, $\{\eta_{tN^{1+\alpha}}(1) : t \geq 0\}$, corresponds to the birth and death process on $\{0, 1/N, \dots, 1\}$ associated to the functions $H(x) = x^\alpha(1-x)^\alpha$, $\lambda_N(x) = (x/x - N^{-1})^\alpha \mathbf{1}\{x \geq 2/N\} + \mathbf{1}\{x = 1/N\}$. In contrast with the set-up of the previous section, λ_N depends on N , but all the arguments go through. Observe that λ_N converges to 1 as $N \uparrow \infty$.

By Theorem 3.1, on the scale $N^{1+\alpha}$, the zero-range process on two sites has a meta-stable behavior with meta-states $\mathcal{E}_N^1 = \{\eta : \eta(1) \geq N - \ell_N\}$, $\mathcal{E}_N^2 = \{\eta : \eta(2) \geq N - \ell_N\}$, and asymptotic Markov evolution with rates $r(1, 2) = r(2, 1) = (m \int_0^1 u^\alpha(1-u)^\alpha du)^{-1}$, where $m = 1 + \sum_{k \geq 1} k^{-\alpha}$.

We now turn to the general case. Recall the definitions introduced in Subsection 2.4. We first estimate the order of $(\theta_N)_{N \geq 1}$. For each $x \in S$ and $N \geq 1$, let the configuration $\xi_N^x \in \mathcal{E}_N^x$ be given by $\xi_N^x(x) = N$. Denote by D_N the Dirichlet form associated to the generator L_N .

In the next lemma and below, we adopt the convention that C_0 stands for a positive finite constant depending only on κ and α whose value may change from line to line.

Lemma 3.2. *There exists a constant $C_0 > 0$ such that*

$$\frac{1}{C_0} < \frac{\theta_N}{N^{1+\alpha}} < C_0$$

for all $N \geq 1$.

Proof. For each $N \geq 1$, we fix the set $I_N := \{0, 1, \dots, N\}$ and the measure

$$\mu_N(n) = \frac{1}{Z_{N,2}} \frac{N^\alpha}{p(N-n)p(n)}, \quad n \in I_N,$$

so that, if $\kappa = 2$, $\nu_N(\eta) = \mu_N(\eta(x))$ for any $x \in \{1, 2\}$.

For $\kappa \geq 2$, given $x, y \in S$, $x \neq y$ and $N \geq 1$, a simple calculation shows that for any $\eta' \in E_N$ such that $\eta'(x) = N - n$ and $\eta'(y) = n$ for some $n \in I_N$ we have

$$\frac{Z_{N,2}}{Z_{N,\kappa}} \mu_N(n) = \nu_N(\eta') \leq \nu_N(\{\eta : \eta(x) = N - n\}) = \frac{Z_{N,2} Z_{n,\kappa-1}}{Z_{N,\kappa}} \mu_N(n).$$

In particular, there exists a constant $C_0 > 0$ such that

$$\frac{1}{C_0} < \frac{\nu_N(\eta')}{\mu_N(n)} \leq \frac{\nu_N(\{\eta : \eta(x) = N - n\})}{\mu_N(n)} < C_0, \quad (3.8)$$

for all $N \geq 1$ and $n \in I_N$.

Fix $x \in S$ and $N \geq 1$. To get a lower bound for $\text{cap}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x)$, fix an arbitrary $f \in \mathcal{B}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x)$ and choose a site $y \in S \setminus \{x\}$. Consider the path $\eta^{(j)}$, $j = 0, 1, \dots, N$ from $\eta^{(0)} = \xi_N^x$ to $\eta^{(N)} = \xi_N^y$ satisfying $\eta^{(j+1)} = \sigma^{x,y} \eta^{(j)}$ for $j = 0, 1, \dots, N-1$. By using Cauchy-Schwarz inequality and (3.8) we obtain

$$\begin{aligned} 1 &= \left\{ \sum_{j=0}^{N-1} [f(\eta^{(j+1)}) - f(\eta^{(j)})] \right\}^2 \\ &\leq D_N(f) \sum_{j=0}^{N-1} \frac{1}{\nu_N(\eta^{(j)}) g(\eta^{(j)}(x))} \\ &\leq D_N(f) \sum_{j=0}^{N-1} \frac{C_0}{\mu_N(j)}. \end{aligned} \quad (3.9)$$

Therefore,

$$\text{cap}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x) \geq C_0 \left\{ \sum_{j=0}^{N-1} \frac{1}{\mu_N(j)} \right\}^{-1} \geq \frac{C_0}{N^{\alpha+1}}.$$

To get a lower bound, consider the function $\varphi_N : I_N \mapsto \mathbb{R}$ defined as $\varphi_N(n) = 0$ for $n \leq \ell_N$, $\varphi_N(n) = 1$ for $n \geq N - \ell_N$ and

$$\varphi_N(n) - \varphi_N(n-1) = \left\{ \sum_{k=\ell_N+1}^{N-\ell_N} \frac{1}{\mu_N(k)g(k)} \right\}^{-1} \frac{1}{\mu_N(n)g(n)}$$

for $\ell_N + 1 \leq n \leq N - \ell_N$. Then, define the function $f_N : E_N \mapsto \mathbb{R}$ by $f_N(\eta) := \varphi_N(\eta(x))$. Hence, $f_N \in \mathcal{B}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x)$ and so $\text{cap}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x)$ is less than or equal to

$$D_N(f_N) = \sum_{n=1}^N [\varphi_N(n) - \varphi_N(n-1)]^2 (\kappa - 1) g(n) \nu_N(\{\eta : \eta(x) = n\}) .$$

By using (3.8) and definition of φ_N we obtain

$$\text{cap}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x) \leq \frac{C_0}{N^{\alpha+1}} .$$

□

We now turn to the

Proof of Theorem 2.5. Since the process is reversible, by Theorem 2.4 and by the definition of θ_N , we just need to verify conditions **(H2)**, **(H3)**.

Since $\ell_N \rightarrow \infty$, by (2.4), $N^{-1}\ell_N \rightarrow 0$. In particular, an elementary computation shows that

$$\lim_{N \rightarrow \infty} \nu_N(\Delta_N) = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} \nu_N(\mathcal{E}_N^x) = 1/\kappa$$

for every $x \in S$. The first identity corresponds to the first assumption of **(H2')**.

For the second claim in **(H2')**, notice that in this setting

$$\partial_x \Delta_N = \{\eta \in E_N : \eta(x) = N - \ell_N - 1\} .$$

For each $x \in S$ and $N \geq 1$ we choose the configuration $\zeta_N^x \in \partial_x \Delta_N$ such that $\zeta_N^x(x) = N - \ell_N - 1$ and $\zeta_N^x(y) = \ell_N + 1$ for some $y \neq x$. It is not difficult to adapt the proof of the lower bound for $\text{cap}_N(\mathcal{E}_N^x, \check{\mathcal{E}}_N^x)$ to show that

$$\text{cap}_N(\zeta_N^x, \check{\mathcal{E}}_N^x) \geq \frac{C_0}{N^{\alpha+1}} .$$

Therefore, the second condition in **(H2')** follows from Lemma 3.2 if one recalls that cap stands for the capacity associated to the generator L_N which has not been speeded up.

Finally, we check hypothesis **(H3)**. In this setting,

$$\partial \mathcal{E}_N^{x,y} = \{\eta \in \mathcal{E}_N : \eta(x) = N - \ell_N\}$$

and

$$\overline{\mathcal{E}_N^x} = \{\eta \in \mathcal{E}_N : \eta(x) \geq N - \ell_N - 1\}$$

for any $x, y \in S$, $x \neq y$ and $N \geq 1$. Fix $x \in S$. It is not difficult to prove that for any $\eta \in \overline{\mathcal{E}_N^x}$ and N large enough

$$\nu_N(\eta) \geq \frac{1}{Z_{N,\kappa}} \frac{1}{(\ell_N + 1)^{\alpha(\kappa-1)}} . \quad (3.10)$$

By similar reasons,

$$\min\{\nu_N(\eta) : \eta \in \partial \mathcal{E}_N^{x,y}\} \geq \frac{C_0}{(\ell_N)^{\alpha(\kappa-1)}} . \quad (3.11)$$

Fix an arbitrary $\eta \in \overline{\mathcal{E}_N^x}$. Consider a path

$$\eta^{(j)} \in \overline{\mathcal{E}_N^x}, \quad j = 0, 1, \dots, p$$

such that $\eta^{(0)} = \eta$, $\eta^{(p)} = \xi_N^x$, $p \leq \ell_N$ and $R_N(\eta^{(j)}, \eta^{(j+1)}) > 0 \forall 0 \leq j < p$. Let f be an arbitrary function in $\mathcal{B}_N(\eta, \xi_N^x)$. By using Cauchy-Schwarz as in (3.9) we obtain

$$D_N(f) \geq \left\{ \sum_{j=0}^{p-1} \frac{1}{\nu_N(\eta^{(j)})} \right\}^{-1}. \quad (3.12)$$

So, it follows from (3.10) and (3.12) that

$$\min \{ \text{cap}_N(\eta, \xi_N^x) : \eta \in \overline{\mathcal{E}_N^x} \} \geq \frac{C_0}{(\ell_N)^{\alpha(\kappa-1)+1}} \quad (3.13)$$

for any $x \in S$. Finally, putting together (3.11) and (3.13), and using Lemma 3.2

$$\theta_N \min \{ \text{cap}_N(\eta, \xi_N^x) : \eta \in \overline{\mathcal{E}_N^x} \} \min \{ \nu_N(\eta) : \eta \in \partial \mathcal{E}_N^{x,y} \} \geq \frac{C_0 N^{\alpha+1}}{(\ell_N)^{2\alpha(\kappa-1)+1}}.$$

Hence, hypotheses **(H3)** follows from assumption (2.4). \square

4. META-STABILITY

In this section we prove Theorem 2.3, Theorem 2.4 and Proposition 2.2.

4.1. Proof of Theorem 2.3. We divide the proof of Theorem 2.3 in two parts: proof of condition (A) and proof of condition (B) in Definition 2.1.

We start with a replacement lemma which is the key point in the proof of condition (A). For a function $V_N : \mathcal{E}_N \mapsto \mathbb{R}$, denote by \hat{V}_N its ν_N -conditional expectation given the σ -algebra generated by the sets $(\mathcal{E}_N^x)_{x \in S}$:

$$\hat{V}_N(\eta) = \frac{1}{\nu_N(\mathcal{E}_N^x)} \sum_{\xi \in \mathcal{E}_N^x} V(\xi) \nu_N(\xi)$$

for any $x \in S$ and η in \mathcal{E}_N^x . Let us also denote

$$\sigma_N(x) := \max_{\eta \in \mathcal{E}_N^x} \mathbb{E}_\eta^\mathcal{E} [H_{\mathcal{E}_N^x}].$$

Lemma 4.1. *Fix a sequence of functions $V_N : \mathcal{E}_N \mapsto \mathbb{R}$ such that*

$$\lim_{N \rightarrow \infty} \sigma_N(x) \max_{\eta \in \mathcal{E}_N^x} |V_N(\eta)| = 0.$$

for any $x \in S$. Then, for any sequence $(\eta_N)_{N \geq 1}$ in \mathcal{E}_N ,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\eta_N}^\mathcal{E} \left[\int_0^t \{V_N(\eta_s^{\mathcal{E}_N}) - \hat{V}_N(\eta_s^{\mathcal{E}_N})\} ds \right] = 0.$$

Proof. Assume without loss of generality that V_N vanishes outside some subset \mathcal{E}_N^x , i.e. $V_N(\eta) = 0$ for $\eta \in \check{\mathcal{E}}_N^x$. By Corollary 5.2, the expectation appearing in the statement of the lemma is bounded by

$$2\sigma_N(x) \|V_N - \hat{V}_N\|_\infty.$$

Since $\|\hat{V}_N\|_\infty \leq \|V_N\|_\infty$ this expression vanishes as $N \uparrow \infty$ by assumption. \square

Condition (A). Fix $x \in S$ and a sequence $(\eta_N)_{N \geq 1}$ such that $\eta_N \in \mathcal{E}_N^x$ for all $N \geq 1$. The convergence of the sequence $\{\mathbb{Q}_{\eta_N}^N\}_{N \geq 1}$ stated in condition (A), follows from tightness and uniqueness of limit points. We first examine the tightness.

Denote by $(\mathbb{P}_\eta^\mathcal{E})_{\eta \in \mathcal{E}_N}$ the trace of the Markov process η_t^N on \mathcal{E}_N , and denote by $\mathbb{E}_\eta^\mathcal{E}$ the expectation with respect to $\mathbb{P}_\eta^\mathcal{E}$.

Lemma 4.2. *The sequence $(\mathbb{Q}_{\eta_N}^N)_{N \geq 1}$ is tight provided conditions **(C1)**, **(C2)** hold.*

Proof. For each $T > 0$, let \mathfrak{T}_T denote the set of all stopping times bounded by T . By Aldous criterium (see Theorem 16.10 in [2]) we just need to show that

$$\lim_{\delta \downarrow 0} \lim_{N \rightarrow \infty} \sup_{\theta \leq \delta} \sup_{\tau \in \mathfrak{T}_T} \mathbb{P}_{\eta_N}^{\mathcal{E}} [|X_{\tau+\theta}^N - X_{\tau}^N| > \epsilon] = 0 \quad (4.1)$$

for every $\epsilon > 0$ and $T > 0$.

Recall that we denote by $L_N^{\mathcal{E}}$ the generator of the trace process $\eta_t^{\mathcal{E}_N}$. Denote by M_t^N the martingale defined by

$$M_t^N = X_t^N - X_0^N - \int_0^t L_N^{\mathcal{E}} X_s^N ds.$$

To prove tightness of the process X_t^N , it is therefore enough to show that (4.1) holds with the difference $X_{\tau+\theta}^N - X_{\tau}^N$ replaced by $M_{\tau+\theta}^N - M_{\tau}^N$ and by $\int_{\tau}^{\tau+\theta} L_N^{\mathcal{E}} X_s^N ds$.

Consider the integral term. By Chebychev inequality and by the strong Markov property, we need to prove that

$$\lim_{\delta \downarrow 0} \lim_{N \rightarrow \infty} \sup_{\theta \leq \delta} \max_{\xi \in \mathcal{E}_N} \mathbb{E}_{\xi}^{\mathcal{E}} \left[\left| \int_0^{\theta} L_N^{\mathcal{E}} X_s^N ds \right| \right] = 0.$$

An elementary computation shows that

$$L_N^{\mathcal{E}} \Psi_N(\eta) = \sum_{x=1}^{\kappa} \{x - \Psi_N(\eta)\} R_N^{\mathcal{E}}(\eta, x)$$

for each η in \mathcal{E}_N . The proof is thus reduced to the claim

$$\lim_{\delta \downarrow 0} \lim_{N \rightarrow \infty} \max_{\xi \in \mathcal{E}_N^y} \mathbb{E}_{\xi}^{\mathcal{E}} \left[\int_0^{\delta} R_N^{\mathcal{E}}(\eta_s, x) ds \right] = 0$$

for all x, y in $\{1, \dots, \kappa\}$.

Fix x, y . In the previous expectation add and subtract the ν_N -conditional expectation of $R_N^{\mathcal{E}}(\eta, x)$ with respect to \mathcal{E}_N^y , denoted by $r_N(y, x)$. By condition **(C1)**, the expectation involving r_N is bounded by $C_0 \delta$, which vanishes as $\delta \downarrow 0$. On the other hand, by condition **(C2)**, $\max_{\eta \in \mathcal{E}_N^y} R_N^{\mathcal{E}}(\eta, x) \max_{\eta \in \mathcal{E}_N^y} \mathbb{E}_{\eta}^N [H_{\xi_N^y}]$ vanishes as $N \uparrow \infty$. Since $\mathbb{E}_{\eta}^{\mathcal{E}} [H_{\xi_N^y}] \leq \mathbb{E}_{\eta}^N [H_{\xi_N^y}]$, we also have that $\max_{\eta \in \mathcal{E}_N^y} R_N^{\mathcal{E}}(\eta, x) \max_{\eta \in \mathcal{E}_N^y} \mathbb{E}_{\eta}^{\mathcal{E}} [H_{\xi_N^y}]$ vanishes as $N \uparrow \infty$. In particular, by Lemma 4.1, the expectation involving the difference $R_N^{\mathcal{E}}(\eta, x) - r_N(y, x)$ vanishes as $N \uparrow \infty$ for every $\delta > 0$.

We now turn to the martingale part M_t^N , whose quadratic variation is given by the time integral of $L_N^{\mathcal{E}} \Psi_N(X_t^N)^2 - 2\Psi_N(X_t^N) L_N^{\mathcal{E}} \Psi_N(X_t^N)$. An elementary computation shows that this expression is equal to

$$\sum_{x=1}^{\kappa} \{x - \Psi_N(\eta)\}^2 R_N^{\mathcal{E}}(\eta, x).$$

Denote by $\langle M^N \rangle_t$ the quadratic variation of the martingale M^N . By the explicit formula for the quadratic variation, by Chebychev inequality and by the strong

Markov property,

$$\begin{aligned} \mathbb{P}_{\eta_N}^{\mathcal{E}} [|M_{\tau+\theta}^N - M_{\tau}^N| > \epsilon] &\leq \frac{1}{\epsilon^2} \mathbb{E}_{\eta_N}^{\mathcal{E}} [\langle M^N \rangle_{\tau+\theta} - \langle M^N \rangle_{\tau}] \\ &\leq \frac{C_0}{\epsilon^2} \max_{\xi \in \mathcal{E}_N} \mathbb{E}_{\xi}^{\mathcal{E}} [\int_0^{\delta} R_N^{\mathcal{E}}(\eta_s, x) ds] \end{aligned}$$

It remains to repeat the arguments presented for the integral term of the decomposition. \square

Now we turn to prove the uniqueness of limit points. Assume without loss of generality that the sequence of probability measures $\mathbb{Q}_{\eta_N}^N$ converges to a measure \mathbb{Q} . Recall definition of $(r_N(x, y) : x, y \in S)$ and their limits $(r(x, y) : x, y \in S)$ in condition **(C1)**. Denote by \mathfrak{L}_N and \mathfrak{L} the Markov generators on the state space $S = \{1, \dots, \kappa\}$ given by

$$(\mathfrak{L}_N F)(x) = \sum_{y \in S \setminus \{x\}} \{F(y) - F(x)\} r_N(x, y)$$

and

$$(\mathfrak{L} F)(x) = \sum_{y \in S \setminus \{x\}} \{F(y) - F(x)\} r(x, y).$$

For $t \geq 0$, let X_t denote the projection $D(\mathbb{R}_+, S) \mapsto S$. We shall prove in Lemma 4.3 below that \mathbb{Q} solves the martingale problem associated to the generator \mathfrak{L} . It is well known that this property together with the distribution of X_0 characterize the measure \mathbb{Q} .

Lemma 4.3. *Assume conditions **(C1)** and **(C2)**. Then, under \mathbb{Q} , $X_0 = x$ and*

$$M_t = F(X_t) - F(X_0) - \int_0^t \mathfrak{L} F(X_s) ds \quad (4.2)$$

is a martingale for any function $F : S \mapsto \mathbb{R}$.

Proof. Fix $0 \leq s < t$, a function $F : S \mapsto \mathbb{R}$ and a bounded function $U : D(\mathbb{R}_+, S) \mapsto \mathbb{R}$ depending only on $\{X_r : 0 \leq r \leq s\}$, continuous for the Skorohod topology. Denote by $\mathbb{E}_{\mathbb{Q}}$ and $\mathbb{E}_{\mathbb{Q}_{\eta_N}^N}$ the expectation with respect to \mathbb{Q} and $\mathbb{Q}_{\eta_N}^N$, respectively. We shall prove that

$$\mathbb{E}_{\mathbb{Q}}[M_t U] = \mathbb{E}_{\mathbb{Q}}[M_s U]. \quad (4.3)$$

Recall that $L_N^{\mathcal{E}}$ denotes the generator of the trace process $\eta_t^{\mathcal{E}_N}$. For $N \geq 1$, consider the $\mathbb{P}_{\eta_N}^{\mathcal{E}}$ -martingale M_t^N defined by

$$M_t^N = F(X_t^N) - F(X_0^N) - \int_0^t L_N^{\mathcal{E}}(F \circ \Psi_N)(\eta_s^{\mathcal{E}_N}) ds.$$

As M_t^N is a martingale, $\mathbb{E}_{\eta_N}^{\mathcal{E}}[M_t^N U] = \mathbb{E}_{\eta_N}^{\mathcal{E}}[M_s^N U]$ so that

$$\mathbb{E}_{\eta_N}^{\mathcal{E}} \left[U \left\{ F(X_t^N) - F(X_s^N) - \int_s^t L_N^{\mathcal{E}}(F \circ \Psi_N)(\eta_r^{\mathcal{E}_N}) dr \right\} \right] = 0.$$

Since $\mathbb{Q}_{\eta_N}^N$ converges to \mathbb{Q} , time averages of $E_{\mathbb{Q}_{\eta_N}^N}[F(X_t)U]$, $E_{\mathbb{Q}_{\eta_N}^N}[F(X_s)U]$ converge to time averages of $E_{\mathbb{Q}}[F(X_t)U]$, $E_{\mathbb{Q}}[F(X_s)U]$, respectively.

We first claim that

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\eta_N}^{\mathcal{E}} \left[U \int_s^t \left\{ (L_N^{\mathcal{E}}(F \circ \Psi_N))(\eta_r^{\mathcal{E}_N}) - (\mathfrak{L}_N F)(X_r^N) \right\} dr \right] = 0. \quad (4.4)$$

Since U is bounded, by the Markov property, it is enough to show that

$$\lim_{N \rightarrow \infty} \max_{\eta \in \mathcal{E}_N} \left| \mathbb{E}_{\eta}^{\mathcal{E}} \left[\int_0^{t-s} \left\{ (L_N^{\mathcal{E}}(F \circ \Psi_N))(\eta_r^{\mathcal{E}_N}) - (\mathfrak{L}_N F)(X_r^N) \right\} dr \right] \right| = 0.$$

By definition of the operator \mathfrak{L}_N and since

$$L_N^{\mathcal{E}}(F \circ \Psi_N)(\eta) = \sum_{y \in S \setminus \{x\}} \{F(y) - F \circ \Psi_N(\eta)\} R_N^{\mathcal{E}}(\eta, y), \quad \text{for } \eta \in \mathcal{E}_N^x,$$

the difference $L_N^{\mathcal{E}}(F \circ \Psi_N)(\eta) - (\mathfrak{L}_N F)(\Psi_N(\eta))$ is equal to

$$\sum_{y \in S} \{F(y) - F(\Psi_N(\eta))\} \{R_N^{\mathcal{E}}(\eta, y) - r_N(\Psi_N(\eta), y)\}.$$

In particular, (4.4) follows from Lemma 4.1 and condition **(C2)** because

$$\mathbb{E}_{\eta}^{\mathcal{E}}[H_{\xi_N^x}] \leq \mathbb{E}_{\eta}^N[H_{\xi_N^x}]$$

for every $\eta \in \mathcal{E}_N^x$. On the other hand, from condition **(C1)**, it follows that

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\eta_N}^{\mathcal{E}} \left[U \int_s^t \left\{ (\mathfrak{L}_N F)(X_r^N) - (\mathfrak{L} F)(X_r^N) \right\} dr \right] = 0.$$

$\mathbb{Q}_{\eta_N}^N$ converges to \mathbb{Q} by assumption. Hence, putting together the previous estimates we obtain that

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\eta_N}^{\mathcal{E}} \left[U \int_s^t (L_N^{\mathcal{E}}(F \circ \Psi_N))(\eta_r^{\mathcal{E}_N}) dr \right] = \mathbb{E}_{\mathbb{Q}} \left[U \int_s^t (\mathfrak{L} F)(X_r) dr \right]$$

because U is bounded and continuous for the Skorohod topology.

Up to this point we proved that

$$\frac{1}{\epsilon} \int_0^{\epsilon} dr E_{\mathbb{Q}} \left[U \left\{ F(X_{t+r}) - F(X_{s+r}) - \int_{s+r}^{t+r} (\mathfrak{L} F)(X_s) \right\} \right] = 0$$

for every $\epsilon > 0$. It remains to let $\epsilon \downarrow 0$ and use the right continuity of the process to deduce (4.3), which concludes the proof of the lemma. \square

Condition (B). We need to introduce some additional notation. For a path $\omega \in D(\mathbb{R}_+, S)$ performing infinitely many jumps, we denote by $\tau_n(\omega)$, $n \geq 0$ the jumping times of ω : $\tau_0(\omega) = 0$ and

$$\tau_n(\omega) := \inf\{t > \tau_{n-1}(\omega) : \omega(t) \neq \omega(\tau_{n-1}(\omega))\}. \quad (4.5)$$

Let us denote

$$T_n(\omega) := \tau_n(\omega) - \tau_{n-1}(\omega), \quad n \geq 1.$$

Further, denote by N_t the number of jumps up to time t :

$$N_t(\omega) := \sup\{j \geq 0 : \tau_j(\omega) \leq t\}.$$

The process $X_t^N := \Psi_N(\eta_t^{\mathcal{E}_N})$ is defined on the path space $D(\mathbb{R}_+, \mathcal{E}_N)$. Nevertheless, X_t^N can be think of as a process defined on $D(\mathbb{R}_+, E_N)$ in the obvious way. Hence, we obtain a coupling of X_t^N and \hat{X}_t^N , where \hat{X}_t^N has been defined just before Proposition 2.2. In what follows we consider this coupling.

Lemma 4.4. *Condition (C3) implies that for every $n \geq 1$*

$$\lim_{N \rightarrow \infty} \max_{\eta \in \mathcal{E}_N} \mathbb{E}_\eta^N [T_n(\hat{X}^N) - T_n(X^N)] = 0. \quad (4.6)$$

Proof. By the strong Markov property, it suffices to prove the limit (4.6) for $n = 1$. Let ω be a path in $D(\mathbb{R}_+, E_N)$. Fix $x \in S$ and $\eta \in \mathcal{E}_N^x$. Under \mathbb{P}_η^N , if $H_{\mathcal{E}_N^x}(\omega) < H_{\Delta_N}(\omega)$ then $T_1(\hat{X}^N(\omega)) = T_1(X^N(\omega))$ and otherwise

$$T_1(\hat{X}^N(\omega)) - T_1(X^N(\omega)) = H_{\mathcal{E}_N^x} \circ \mathcal{R}^{E_N \setminus \mathcal{E}_N^x} \circ \Theta(H_{\Delta_N})(\omega) \quad (4.7)$$

where $\Theta(H_{\Delta_N})(\omega) \in D(\mathbb{R}_+, E_N)$ is defined by

$$\Theta(H_{\Delta_N})(\omega)(t) = \omega(H + t)$$

and $H := H_{\Delta_N}(\omega)$. By using (4.7) and the strong Markov property,

$$\begin{aligned} \mathbb{E}_\eta^N [T_1(\hat{X}^N) - T_1(X^N)] &\leq \max_{\xi \in \partial_x \Delta_N} \mathbb{E}_\xi^N [H_{\mathcal{E}_N^x} \circ \mathcal{R}^{E_N \setminus \mathcal{E}_N^x}] \\ &= \max_{\xi \in \partial_x \Delta_N} \mathbb{E}_\xi^{N,x} [H_{\mathcal{E}_N^x}] \end{aligned}$$

Hence, (4.6) for $n = 1$ follows from condition (C3). \square

Now, we proceed to check condition (B). To keep notation simple, set $\hat{N}_t := N_t(\hat{X}^N)$, $\hat{T}_n := T_n(\hat{X}^N)$ and $T_n := T_n(X^N)$. Fix an arbitrary $\eta \in \mathcal{E}_N$. Under \mathbb{P}_η^N , it is easy to see that

$$\begin{aligned} \mathcal{T}_t^{\Delta_N} &\leq t \wedge \sum_{n=1}^{\hat{N}_t+1} (\hat{T}_n - T_n) \\ &\leq \mathbf{1}\{\hat{N}_t \geq K\} t + \mathbf{1}\{\hat{N}_t < K\} \sum_{n=1}^K (\hat{T}_n - T_n), \end{aligned}$$

for any positive integer K . Therefore,

$$\mathbb{E}_\eta^N [\mathcal{T}_t^{\Delta_N}] \leq t \mathbb{P}_\eta^N [N_t(\hat{X}^N) \geq K] + \mathbb{E}_\eta^N [\tau_K(\hat{X}^N) - \tau_K(X^N)]. \quad (4.8)$$

It follows from the previous lemma that

$$\lim_{N \rightarrow \infty} \max_{\eta \in \mathcal{E}_N} \mathbb{E}_\eta^N [\tau_K(\hat{X}^N) - \tau_K(X^N)] = 0.$$

Hence, in view of estimate (4.8), it remains to prove that

$$\lim_{K \rightarrow \infty} \limsup_{N \rightarrow \infty} \max_{\eta \in \mathcal{E}_N} \mathbb{P}_\eta^N [N_t(\hat{X}^N) \geq K] = 0. \quad (4.9)$$

For it, notice that $N_t(\hat{X}^N) \leq N_t(X^N)$, \mathbb{P}_η^N - a.s. for all $\eta \in \mathcal{E}_N$ and, in consequence,

$$\mathbb{P}_\eta^N [N_t(\hat{X}^N) \geq K] \leq \mathbb{P}_\eta^N [N_t(X^N) \geq K].$$

Fix the sequence $\eta_N \in \mathcal{E}_N$, $N \geq 1$ defined by

$$\begin{aligned} \max_{\eta \in \mathcal{E}_N} \mathbb{P}_\eta^N [N_t(X^N) \geq K] &:= \mathbb{P}_{\eta_N}^N [N_t(X^N) \geq K] \\ &= \mathbb{Q}_{\eta_N}^N [N_t \geq K]. \end{aligned} \quad (4.10)$$

Now, it is not difficult to see that $[N_t \geq K]$ is a closed set for the Skorohod topology on $D(\mathbb{R}_+, S)$. Then,

$$\limsup_{N \rightarrow \infty} \mathbb{Q}_{\eta_N}^N [N_t \geq K] \leq \max_{x \in S} \mathbb{Q}_x [N_t \geq K]. \quad (4.11)$$

Finally, since S is a finite set, limit (4.9) follows from (4.10) and (4.11). This concludes the proof of Theorem 2.3.

4.2. Proof of Theorem 2.4. Recall that ν_N is supposed to be a reversible probability measure for all $N \geq 1$. We recall the formula for the expected value of entry times proved in [4] : For any subset F of E_N and any $\eta \in E_N \setminus F$,

$$\mathbb{E}_\eta^N[H_F] = \frac{\nu_N(f_{\eta,F})}{\text{cap}_N(\eta, F)}. \quad (4.12)$$

This is the key result in order to introduce the notion of capacity in our study of meta-stability.

In what follows, we check each condition in Theorem 2.3. Obviously, condition **(C1)** is already in force. To check condition **(C2)**, fix two elements x, y of S , $x \neq y$. By using the strong Markov property, we have for any two sites $\eta, \xi \in \overline{\mathcal{E}_N^x}$,

$$\mathbb{E}_\eta^N[H_\xi] \leq \mathbb{E}_\eta^N[H_{\xi_N^x}] + \mathbb{E}_{\xi_N^x}^N[H_\xi].$$

Then, by using formula (4.12),

$$\max_{\eta, \xi \in \overline{\mathcal{E}_N^x}} \mathbb{E}_\eta^N[H_\xi] \leq \frac{2}{\min\{\text{cap}_N(\eta, \xi_N^x) : \eta \in \overline{\mathcal{E}_N^x}\}}. \quad (4.13)$$

On the other hand, if $\partial\mathcal{E}_N^{x,y}$ is empty then $\max_{\eta \in \mathcal{E}_N^x} R_N^\mathcal{E}(\eta, y) = 0$. Otherwise, an elementary estimate shows that

$$\max_{\eta \in \mathcal{E}_N^x} R_N^\mathcal{E}(\eta, y) \leq \frac{\nu_N(\mathcal{E}_N^x) r_N(x, y)}{\min\{\nu_N(\eta) : \eta \in \partial\mathcal{E}_N^{x,y}\}}. \quad (4.14)$$

Therefore, condition **(C2)** follows from (4.13), (4.14) and hypotheses **(H3)** because $r_N(x, y)$ is bounded by virtue of hypotheses **(C1)**.

Finally we check condition **(C3)**. Fix $x \in S$. By using the strong Markov property for the trace of η_t^N on $E_N \setminus \mathcal{E}_N^x$,

$$\max_{\eta \in \partial_x \Delta_N} \mathbb{E}_\eta^{N,x}[H_{\mathcal{E}_N^x}] \leq \max_{\eta \in \partial_x \Delta_N} \mathbb{E}_\eta^{N,x}[H_{\zeta_N^x}] + \mathbb{E}_{\zeta_N^x}^{N,x}[H_{\tilde{\mathcal{E}}_N^x}].$$

By hypotheses **(H3)** and estimate (4.13), the first term in the right hand side of this inequality vanishes as $N \uparrow \infty$, because $\mathbb{E}_\eta^{N,x}[H_{\zeta_N^x}] \leq \mathbb{E}_\eta^N[H_{\zeta_N^x}]$ for all $\eta \in \partial_x \Delta_N$. Moreover, by using the formula in (e) of Lemma 5.4,

$$\begin{aligned} \mathbb{E}_{\zeta_N^x}^{N,x}[H_{\tilde{\mathcal{E}}_N^x}] &= \frac{\nu_N(\mathbf{1}_{\{E_N \setminus \mathcal{E}_N^x\}} f_{\zeta_N^x, \tilde{\mathcal{E}}_N^x})}{\text{cap}_N(\zeta_N^x, \tilde{\mathcal{E}}_N^x)} \\ &\leq \frac{\nu_N(\Delta_N)}{\text{cap}_N(\zeta_N^x, \tilde{\mathcal{E}}_N^x)}. \end{aligned}$$

In view of **(H2)**, this concludes the proof.

4.3. Proof of Proposition 2.2. At first, we fix a metric in $D(\mathbb{R}_+, S)$ which induces the Skorohod topology. For each integer $m \geq 1$, let Λ_m denote the class of strictly increasing, continuous mappings of $[0, m]$ onto itself. If $\lambda \in \Lambda$, then $\lambda_0 = 0$ and $\lambda_m = m$. For ω and $\hat{\omega}$ in $D(\mathbb{R}_+, S)$, define $d_m(\omega, \hat{\omega})$ to be the infimum of those positive ϵ for which there exists in Λ_m a λ satisfying

$$\sup_{t \in [0, m]} |\lambda_t - t| < \epsilon$$

and

$$\sup_{t \in [0, m]} |\omega(\lambda_t) - \hat{\omega}(t)| < \epsilon .$$

Now, for each integer $m \geq 1$, define

$$g_m(t) = \begin{cases} 1 & \text{if } t \leq m-1, \\ m-t & \text{if } m-1 \leq t \leq m, \\ 0 & \text{if } t \geq m. \end{cases}$$

For each $\omega \in D(\mathbb{R}_+, S)$ let ω^m the element of $D(\mathbb{R}_+, S)$ defined by

$$\omega^m(t) = g_m(t)\omega(t), \quad t \geq 0 .$$

Finally, we define the metric in $D(\mathbb{R}_+, S)$ by

$$d(\omega, \hat{\omega}) = \sum_{m=1}^{\infty} 2^{-m} (1 \wedge d_m(\omega^m, \hat{\omega}^m)) .$$

This metric induces the Skorohod topology in $D(\mathbb{R}_+, S)$ (cf. [2]).

Recall the jumping times $\tau_n(\omega)$, $n \geq 0$ defined in (4.5) for any path $\omega \in D(\mathbb{R}_+, S)$ performing infinitely many jumps. Further, let $n_t(\omega)$ stand for the number of jumps of ω strictly before time $t > 0$:

$$n_t(\omega) := \sup\{j \geq 0 : \tau_j(\omega) < t\} .$$

Now, let ω and $\hat{\omega}$ be elements of $D(\mathbb{R}_+, S)$ which perform infinitely many jumps. To keep notation simple set $\tau_n := \tau_n(\omega)$, $\hat{\tau}_n := \tau_n(\hat{\omega})$ and $\hat{n}_t := n_t(\hat{\omega})$. Suppose that

$$\hat{\tau}_{n+1} - \hat{\tau}_n \geq \tau_{n+1} - \tau_n \quad \text{and} \quad \omega(\tau_n) = \hat{\omega}(\hat{\tau}_n) \quad \text{for all } n \geq 0, \quad (4.15)$$

so that $\omega(s) = \hat{\omega}(t)$ for $(s, t) \in [\tau_n, \tau_{n+1}[\times [\hat{\tau}_n, \hat{\tau}_{n+1}[$ and $n \geq 0$. Then, we claim that

$$d_m(\omega^m, \hat{\omega}^m) \leq \kappa \max\{\hat{\tau}_{\hat{n}_m} - \tau_{\hat{n}_m}, (m - \tau_{\hat{n}_m+1})^+\} \quad (4.16)$$

for all integer $m \geq 1$. Let us prove the claim. Notice that $\tau_{\hat{n}_m} < m$ for every $m \geq 1$. Choose $\epsilon > 0$ small enough such that $\hat{\tau}_{\hat{n}_m} < m - \epsilon$ and $\tau_{\hat{n}_m} < (m \wedge \tau_{\hat{n}_m+1}) - \epsilon$. Now, let $\lambda \in \Lambda_m$ be given by: $\lambda_{\hat{\tau}_j} = \tau_j$, for $j \leq \hat{n}_m$, $\lambda_{m-\epsilon} = (m \wedge \tau_{\hat{n}_m+1}) - \epsilon$, $\lambda_m = m$ and we complete λ on $[0, m]$ by linear interpolation. Then,

$$\sup_{t \in [0, m]} |\lambda_t - t| \leq \max\{\hat{\tau}_{\hat{n}_m} - \tau_{\hat{n}_m}, (m - \tau_{\hat{n}_m+1})^+\} .$$

Moreover,

$$\begin{aligned} \sup_{t \in [0, m-\epsilon]} |g(\lambda_t)\omega(\lambda_t) - g(t)\hat{\omega}(t)| &\leq \kappa \sup_{t \in [0, m-\epsilon]} |g(\lambda_t) - g(t)| \\ &\leq \kappa \max\{\hat{\tau}_{\hat{n}_m} - \tau_{\hat{n}_m}, (m - \tau_{\hat{n}_m+1})^+\} \end{aligned}$$

and

$$\begin{aligned} \sup_{t \in [m-\epsilon, m]} |g(\lambda_t)\omega(\lambda_t) - g(t)\hat{\omega}(t)| &\leq \kappa \sup_{t \in [m-\epsilon, m]} (|g(\lambda_t)| + |g(t)|) \\ &\leq \kappa \{(m - \tau_{\hat{n}_m+1})^+ + 2\epsilon\} . \end{aligned}$$

Since ϵ is arbitrary small, the claim is proved.

Recall the coupling of \hat{X}_t^N and X_t^N used in the proof of Condition (B). Then, in order to prove Proposition 2.2, it is enough to prove that for any $\delta > 0$

$$\lim_{N \rightarrow \infty} \mathbb{P}_{\eta_N}^N [d(X^N, \hat{X}^N) > \delta] = 0 .$$

Notice that the paths of \hat{X}_t^N and X_t^N in the coupling satisfy conditions (4.15) and, in consequence, estimate (4.16). Let us denote $\hat{n}_m^N := n_m(\hat{X}^N)$. To estimate the right hand side in (4.16), observe that

$$\tau_{\hat{n}_m^N}(\hat{X}^N) - \tau_{\hat{n}_m^N}(X^N) = \mathcal{T}_{\hat{n}_m^N}^{\Delta_N} \leq \mathcal{T}_m^{\Delta_N}$$

and $(m - \tau_{\hat{n}_m^N+1}(X^N))^+ \leq \mathcal{T}_m^{\Delta_N}$. Therefore, by (4.16) we have

$$d(X^N, \hat{X}^N) \leq \sum_{m=1}^{\infty} 2^{-m} (1 \wedge \kappa \mathcal{T}_m^{\Delta_N}).$$

From this estimate and condition (B) of meta-stability it follows that

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\eta_N}^N [d(X^N, \hat{X}^N)] = 0.$$

This completes the proof of Proposition 2.2.

5. MARKOV PROCESSES

We state in this section several properties of Markov processes used throughout the article. Consider an irreducible Markov process $\{\eta_t : t \geq 0\}$ on a finite state space E and denote its unique invariant measure by ν . Denote by \mathbb{P}_η the probability measure on the path space $D(\mathbb{R}_+, E)$ corresponding to the Markov process η_t starting from η and let \mathbb{E}_η stand for the expectation with respect to \mathbb{P}_η .

5.1. Replacement Lemma. In this subsection we prove the result used in Lemma 4.1. We start with the following identity.

Lemma 5.1. *Fix a mean-zero function $V : E \rightarrow \mathbb{R}$ and a site η in E . Let H_η be the entry time to η : $H_\eta = \inf\{t \geq 0 : \eta_t = \eta\}$. Then, for all ξ in E ,*

$$\mathbb{E}_\xi \left[\int_0^t V(\eta_s) ds \right] = \mathbb{E}_\xi \left[\int_0^{H_\eta} V(\eta_s) ds \right] - \mathbb{E}_\xi \left[\mathbb{E}_{\eta_t} \left[\int_0^{H_\eta} V(\eta_s) ds \right] \right].$$

Proof. Let $(\Theta_t)_{t \geq 0}$ stand for the shift operators. Define $H_0 = 0$,

$$H_1 = \inf\{t \geq 0 : \eta_t = \eta, \eta_s \neq \eta \text{ for some } 0 \leq s < t\}$$

and $H_{j+1} = H_1 \circ \Theta_{H_j}$ for $j \geq 1$. The expected value appearing in the statement of the lemma can be written as

$$\begin{aligned} & \sum_{j \geq 0} \mathbb{E}_\xi \left[\int_0^t V(\eta_s) ds \mathbf{1}\{H_j \leq t < H_{j+1}\} \right] \\ &= \sum_{j \geq 0} \sum_{k=0}^j \mathbb{E}_\xi \left[\int_{H_k}^{H_{k+1}} V(\eta_s) ds \mathbf{1}\{H_j \leq t < H_{j+1}\} \right] \\ &- \sum_{j \geq 0} \mathbb{E}_\xi \left[\int_t^{H_{j+1}} V(\eta_s) ds \mathbf{1}\{H_j \leq t < H_{j+1}\} \right]. \end{aligned} \tag{5.1}$$

We compute these two terms separately. The first one can be rewritten as

$$\sum_{k \geq 0} \mathbb{E}_\xi \left[\int_{H_k}^{H_{k+1}} V(\eta_s) ds \mathbf{1}\{H_k \leq t\} \right].$$

Taking conditional expectation with respect to \mathcal{F}_{H_k} , by the strong Markov property, this sum is equal to

$$\mathbb{E}_\xi \left[\int_0^{H_1} V(\eta_s) ds \right] + \mathbb{E}_\eta \left[\int_0^{H_1} V(\eta_s) ds \right] \sum_{k \geq 1} \mathbb{P}_\xi [H_k \leq t] .$$

Notice that in the second expectation, η appears instead of ξ . Since V has mean-zero with respect to the unique stationary measure ν , the second expectation vanishes.

We now turn to the second term in the decomposition (5.1). Observe that $\{H_j \leq t < H_{j+1}\}$ belongs to the σ -algebra \mathcal{F}_t and on this set $H_{j+1} = t + H_1 \circ \Theta_t$. Therefore, by the Markov property, the second term in (5.1) is equal to

$$\sum_{j \geq 0} \mathbb{E}_\xi \left[\mathbf{1}_{\{H_j \leq t < H_{j+1}\}} \mathbb{E}_{\eta_t} \left[\int_0^{H_1} V(\eta_s) ds \right] \right] = \mathbb{E}_\xi \left[\mathbb{E}_{\eta_t} \left[\int_0^{H_1} V(\eta_s) ds \right] \right] .$$

We have just proved that

$$\mathbb{E}_\xi \left[\int_0^t V(\eta_s) ds \right] = \mathbb{E}_\xi \left[\int_0^{H_1} V(\eta_s) ds \right] - \mathbb{E}_\xi \left[\mathbb{E}_{\eta_t} \left[\int_0^{H_1} V(\eta_s) ds \right] \right] .$$

On the one hand, if $\xi \neq \eta$ then $H_\eta = H_1$ \mathbb{P}_ξ -a.s. and so

$$E_\xi \left[\int_0^{H_1} V(\eta_s) ds \right] = E_\xi \left[\int_0^{H_\eta} V(\eta_s) ds \right] .$$

On the other hand, if $\xi = \eta$ then $H_\eta = 0$ \mathbb{P}_ξ -a.s. and so

$$E_\eta \left[\int_0^{H_1} V(\eta_s) ds \right] = 0 = E_\eta \left[\int_0^{H_\eta} V(\eta_s) ds \right] .$$

This concludes the proof of the lemma. \square

Corollary 5.2. *Assume further that V vanishes outside a subset F of E . Then,*

$$\max_{\xi \in E} \left| \mathbb{E}_\xi \left[\int_0^t V(\eta_s) ds \right] \right| \leq 2 \|V\|_\infty \max_{\xi \in F} \mathbb{E}_\xi [H_\eta] .$$

Proof. We need to estimate the two expectations appearing on the right hand side of the statement of Lemma 5.1. Recall that H_F denotes the entry time in F defined in (2.1). Since V vanishes outside F , in the first expectation we may take the time integral on the interval $[H_F, H_\eta]$ and apply the strong Markov property to rewrite this term as

$$\mathbb{E}_\xi \left[\mathbb{E}_{\eta_{H_F}} \left[\int_0^{H_\eta} V(\eta_s) ds \right] \right] .$$

This expression is absolutely bounded by $\|V\|_\infty \max_{\xi \in F} \mathbb{E}_\xi [H_\eta]$. By the same reasons, the second term is bounded above by the same quantity. \square

5.2. The trace process. We recall in this subsection some elementary properties of trace processes and deduce from these attributes some identities used throughout the article.

Denote by L the generator of the Markov process η_t and by $R(\cdot, \cdot)$ its transition rates so that

$$(Lf)(\eta) = \sum_{\xi \in E} \{ f(\xi) - f(\eta) \} R(\eta, \xi)$$

for all functions $f : E \rightarrow \mathbb{R}$. Let $p(\cdot, \cdot)$ be the jump probabilities and λ the holding times:

$$\lambda(\eta) = \sum_{\xi \neq \eta} R(\eta, \xi), \quad p(\eta, \xi) = \frac{1}{\lambda(\eta)} R(\eta, \xi).$$

Fix a non-empty subset F of E . Denote by η_t^F the trace process of η_t on F . Denote by L^F the generator of the Markov process η_t^F , by $R^F(\cdot, \cdot)$ its transition rates, by λ^F the holding times and by $p^F(\cdot, \cdot)$ the jump probabilities. We have that

$$\lambda^F(\eta) = \lambda(\eta) \mathbb{P}_\eta[H_{F \setminus \{\eta\}} < \tau_\eta], \quad p^F(\eta, \xi) = \mathbb{P}_\eta[H_{\{\xi\}} = H_{F \setminus \{\eta\}}] \quad (5.2)$$

for $\eta, \xi \in F$, $\eta \neq \xi$, where τ_η stands for the time of the first return to η :

$$\tau_\eta = \inf \{t \geq 0 : \eta_t = \eta, \eta_s \neq \eta \text{ for some } 0 \leq s < t\}.$$

To prove the first identity in (5.2), denote by τ_1^F the first jumping time of the process η_t^F . Under \mathbb{P}_η , τ_1^F can be represented as

$$\tau_1^F = \sum_{j=1}^N S_j,$$

where S_j are i.i.d. exponential random variables with parameter $\lambda(\eta)$ and, by the strong Markov property, N is a geometric r.v. independent of the sequence $\{S_j : j \geq 1\}$ and such that $\mathbb{P}_\eta[N = k] = a(1 - a)^{k-1}$, $k \geq 1$, for $a = \mathbb{P}_\eta[H_{F \setminus \{\eta\}} < \tau_\eta]$. It remains to recall that an independent geometric sum of independent exponential r.v. is exponentially distributed. The second identity in (5.2) is obvious.

An explicit formula for λ^F , p^F can be obtained recursively. Indeed, an elementary computation shows that, when $F = E \setminus \{\xi_0\}$ for some ξ_0 in E ,

$$R^F(\eta, \xi) = R(\eta, \xi) + R(\eta, \xi_0) p(\xi_0, \xi) \quad (5.3)$$

for $\eta \neq \xi$, $\{\eta, \xi\} \subseteq E \setminus \{\xi_0\}$.

From now on, we assume that η_t is an irreducible Markov process. Denote by ν its unique invariant measure.

Lemma 5.3. *For any non-empty subset F of E , $\{\eta_t^F : t \geq 0\}$ is an irreducible Markov process. The measure ν conditioned to F , denoted by ν^F , is the unique invariant probability measure of η_t^F . Moreover, ν^F is reversible, whenever ν is.*

Proof. For $F = E \setminus \{\xi_0\}$, the lemma follows from identity (5.3) and straightforward computations. To prove it for general subsets F we proceed by induction on the cardinality of F . \square

For each $F \subseteq E$, and all subsets G_1, G_2 of F such that $G_1 \cap G_2 = \emptyset$ we denote

$$r_F(G_1, G_2) := \frac{1}{\nu(G_1)} \sum_{\substack{\eta \in G_1 \\ \xi \in G_2}} \nu(\eta) R^F(\eta, \xi).$$

Observe that if ν is reversible, then

$$\nu(G_1) r_F(G_1, G_2) = \nu(G_2) r_F(G_2, G_1).$$

Recall the definition of the capacity of irreducible reversible Markov processes introduced in Subsection 2.3. We denote by $\text{cap}(\cdot, \cdot)$, respectively by $\text{cap}_F(\cdot, \cdot)$, the capacity for η_t , respectively for η_t^F .

We summarize in the next lemma, properties of the process η_t^F used throughout Section 4. For each η in F , let \mathbb{P}_η^F be the probability measure on the path space

$D(\mathbb{R}_+, F)$ induced by the Markov process η_t^F starting from η . Also, denote by \mathbb{E}_η^F the expectation with respect to \mathbb{P}_η^F for each $\eta \in F$.

Lemma 5.4. *Fix a non-empty subset F of E , and two subsets G_1, G_2 of F such that $G_1 \cap G_2 = \emptyset$. There holds,*

(a) *For all η in F ,*

$$\mathbb{P}_\eta^F[H_{G_1} < H_{G_2}] = \mathbb{P}_\eta[H_{G_1} < H_{G_2}].$$

(b) *Let $\check{G}_2 = F \setminus G_2$, then*

$$\nu(G_1) r_F(G_1, G_2) = \sum_{\substack{\eta \in G_1 \\ \xi \in E}} \nu(\eta) R(\eta, \xi) \mathbb{P}_\xi[H_{G_2} < H_{\check{G}_2}].$$

(c) *If ν is reversible and $F = G_1 \cup G_2$, then*

$$\nu(G_1) r_F(G_1, G_2) = \text{cap}(G_1, G_2).$$

(d) *If ν is reversible, then*

$$\text{cap}_F(G_1, G_2) = \frac{\text{cap}(G_1, G_2)}{\nu(F)}.$$

(e) *If ν is reversible, then for any $\eta \in F \setminus G$*

$$\mathbb{E}_\eta^F[H_G] = \frac{\sum_{\xi \in F} \mathbb{P}_\xi[H_{\{\eta\}} < H_G] \nu(\xi)}{\text{cap}(\{\eta\}, G)}.$$

Proof. Claim (a) is obvious from the construction of the trace process. To prove Claim (b), observe that we may write the left hand side of (b) as

$$\sum_{\substack{\eta \in G_1 \\ \xi \in F}} \nu(\eta) R^F(\eta, \xi) \mathbb{P}_\xi^F[H_{G_2} < H_{\check{G}_2}].$$

Enumerate the set $E \setminus F$ as $\{\xi_1, \dots, \xi_M\}$. Let $E_0 = \emptyset$, $E_i = \{\xi_1, \dots, \xi_i\}$, for $1 \leq i \leq M$. We prove by induction that the previous expression is equal to

$$\sum_{\substack{\eta \in G_1 \\ \xi \in F \cup E_i}} \nu(\eta) R^{F \cup E_i}(\eta, \xi) \mathbb{P}_\xi^{F \cup E_i}[H_{G_2} < H_{\check{G}_2}] \quad (5.4)$$

for $0 \leq i \leq M$. Claim (b) follows from this identity since the right hand side of (b) coincide with the previous sum for $i = M$.

Obviously, the two previous sums are equal for $i = 0$. Assume that the identity holds for some $0 \leq i \leq M - 1$. By (5.3), (5.4) is equal to

$$\begin{aligned} & \sum_{\substack{\eta \in G_1 \\ \xi \in F \cup E_i}} \nu(\eta) R^{F \cup E_{i+1}}(\eta, \xi) \mathbb{P}_\xi^{F \cup E_i}[H_{G_2} < H_{\check{G}_2}] \\ & + \sum_{\substack{\eta \in G_1 \\ \xi \in F \cup E_i}} \nu(\eta) R^{F \cup E_{i+1}}(\eta, \xi_{i+1}) p^{F \cup E_{i+1}}(\xi_{i+1}, \xi) \mathbb{P}_\xi^{F \cup E_i}[H_{G_2} < H_{\check{G}_2}]. \end{aligned}$$

By (a), we may replace in the previous formula $F \cup E_i$ by $F \cup E_{i+1}$ in the superscript of \mathbb{P}_ξ . After this replacement, note that

$$\sum_{\xi \in F \cup E_i} p^{F \cup E_{i+1}}(\xi_{i+1}, \xi) \mathbb{P}_\xi^{F \cup E_{i+1}}[H_{G_2} < H_{\check{G}_2}] = \mathbb{P}_{\xi_{i+1}}^{F \cup E_{i+1}}[H_{G_2} < H_{\check{G}_2}]$$

since the left hand side is obtained from the right hand side by conditioning on the first jump. This proves (b).

To prove claims (c), (d) and (e), for the remaining part of the proof we suppose that ν is reversible. By arguments similar to the proof of Theorem II.6.1 in [12], we have

$$\text{cap}(G_1, G_2) = \sum_{\substack{\eta \in G_1 \\ \xi \in E}} \nu(\eta) R(\eta, \xi) \mathbb{P}_\xi [H_{G_2} < H_{G_1}] . \quad (5.5)$$

Therefore, claim (c) follows from this identity and claim (b) when $F = G_1 \cup G_2$. Further, it follows from the proof of (b) that

$$\sum_{\substack{\eta \in G_1 \\ \xi \in F}} \nu(\eta) R^F(\eta, \xi) \mathbb{P}_\xi^F [H_{G_2} < H_{G_1}] = \sum_{\substack{\eta \in G_1 \\ \xi \in E}} \nu(\eta) R(\eta, \xi) \mathbb{P}_\xi [H_{G_2} < H_{G_1}] .$$

By the version of identity (5.5) for the trace process η_t^F , the left hand side of this equation is equal to $\nu(F) \text{cap}_F(G_1, G_2)$. By (5.5), this completes the proof of (d). Finally, claim (e) follows easily from formula (4.12) for the trace process η_t^F , claim (a) and claim (d). We are done. \square

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